International Journal of Theoretical Physics, Vol. 4, No. 6 (1971), pp. 427–431

Could Classical States also be Distributions with Extended Supports?

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Received 1 *April* 1971

Abstract

Unlike states in quantum mechanics or kinetic theory, it is generally believed that a classical state is a function with no spread; i.e., it does not involve any statistics. But a unified theory of dynamical processes suggests the plausibility of substituting for it the functional of error distribution.

Introduction

The essential specialness of a quantum measurement is that it is impossible to obtain simultaneously sharply defined values for both position and momentum variables. Quantitatively, this means that if a state is carefully prepared for which the position has a variance tending to zero, then the momentum of that state has a variance which tends to infinity. Thus in general, a quantum observable has only a probability distribution of values and not a sharply defined one. Hence, one restricts the logic of the physical statements about a quantum system to a subset of those which are experimentally verifiable (Birkhoff & von Neumann, 1936). It is further proved that this subset constitutes a projective geometry and not a Boolean algebra. This situation is in contradistinction to the classical case where points in the phase space Z are correlated with experimental propositions in a one-to-one correspondence. This implies (Sudarshan, 1961) that the classical state is a pure state corresponding to the extreme points of the convex set of Z. In kinetic theory, at least Lebesgue-measurable subsets of Z are kept in correspondence with the experimental propositions; but in classical mechanics the stronger assumption, that functions on Z represent the dynamical variables of the system, implies that the pure state in question is also the δ-distribution of point support—as opposed to a statistical case where the support has an extension in Z. Correspondingly, the elements of the logic of kinetic theory are Lebesque-measurable subsets of Z and those of classical mechanics cover all the points of Z. This leads one to believe that the numerical result of a measurement of an observable in classical mechanics is identical with the function representing it! As a consequence, the measurement of a product of classical observables is also set equal to the product of their measurements, whereas, in the quantum case, there is generally a dispersion. It is precisely this distinction which makes it impossible to quantise classical motion in a structure-preserving way (Arens & Babbitt, 1965). By this we mean that the transformation of a product of classical observables into operators with specified commutators can never be equal to the product of their transformations. (The product of classical observables referred to here is the point product.) But the profound algebraic equivalence of the classical and quantum theories (Jordan & Sudarshan, 1961) would lose much of its viability with the impossibility of structure preservation. One would naturally ask, 'If structure is not preserved, what then is the extent to which the two theories are equivalent?'

However, a more detailed analysis of the problem of structure preservation, so as to preserve also the algebraic equivalence, leads to the following conclusion: namely, classical observables of an equivalent description of quantum theory cannot be considered as functions on Z, but must be deemed to be functionals (Shankara & Srinivas, 1971) acting over suitable distributions on Z, just as in a statistical theory. This conclusion, arrived at in a devious way through an overall study of the structure of dynamics is, after all, also our daily experience. In practice, a classical measurement is the average of several readings which are theoretically infinite in number. In the language of analysis, the 'objective' result of a measurement is only the limit of a sequence of measurements (Ghosh, 1957). Now it is also a fact that not all readings are the same; apart from chance repetitions, the readings do range about a certain mean value. From where do these differences arise? A statistics appears to be inexorably linked up with the process. This can mean that the 'status' of a classical system (Sudarshan, 1961) may not in reality be a δ-function, but a distribution with finite extended support in Z.

But the important difference between this statistics and the one occurring in quantum measurements should definitely not be lost sight of: in the latter there exists no state in which both position and momentum have simultaneously a small variance; i.e., there is no Boolean algebra for which the sublogics of position and momentum are Boolean subalgebras. On the other hand, in the classical case, there is no such correlation between conjugate observables and for *any observable* the variance could be small, *though not strictly zero*. What is envisaged in the foregoing analysis is hence only the inescapable 'subjective' statistics arising due to the interaction of the classical system with the measuring apparatus. For the purposes of this article, a classical state would mean that state of an event which lies entirely within the range of human perception. Hence, we search for this statistics in the physiology of the eye or the ear. Although these arguments, with suitable modifications, must be applicable to any generic observer, we choose the human physiology because, ultimately, all measurements are to be traced to the human observer (von Neumann, 1955).

1. *Visual Measurements*

This process consists in reading a point against a given mark on a reference scale. In order to improve the accuracy in reading, a vernier is generally attached to the scale. The principle involved (Parson, 1964) is that the eye has the visual capacity to distinguish any irregularity in the line of demarcation between two contours which is of the order of 5 sec of arc. (The clinical standard is actually 1 min.) This sense of discrimination has this lower limit because of the obvious limitations on the lens system of the eye. This is the optimum that can be achieved between resolving power and illumination with a normal pupillary diameter of 2 mm by which 70% of the light is utilised. But the pupillary aperture itself executes rapid oscillations the moment light falls on it. It first contracts, then oscillates rapidly, and finally settles down into a state of contraction. This is not any pathological condition, but the situation in the normal eye. So there is no way of knowing for certain at what instant in this duration the reading is to be made. Besides these oscillations, there are still other pupillary reflexes of a random nature. One that can be mentioned in this context is the so-called psycho-sensory reflex, initiated by the stimulation of any sensory nerve or by emotional states and excitement. The pupillary response is a measure of the interest, emotion, thought processes and attitudes of the observer (Hess, 1965)! This is a very complicated movement, controlled by the two muscles—the sphincter and the dilator—attached to the pupil. The responses of these muscles to excitement are very rapid and delicate, and the instantaneous size of the pupil is essentially the result of their opposing forces.

2. *Auditory Measurements*

The element of randomness in this process is more gross because longer wavelengths are involved. The smallest change in frequency the ear (Scott-Brown, *et al.*, 1965) can detect is different at different intensity levels, and this sensitivity is greatest in the range of 2000–3000 c/sec which is actually equal to a sound pressure of 2×10^{-4} dyn/cm^2. This is 15–30 dB higher than the intensity of the ubiquitous ambient sound energy due to the Brownian motion of the so-called cochlear fluids in the inner ear. Indeed, increased acuity beyond this limit is a liability.

In the middle ear there are two suitably placed windows, which finally let into the inner ear a mixture of the sound wave and the wave with its phase reversed. This superposition gives a to-and-fro motion to the cochlear fluids as a whole, without causing any significant compression or rarefaction of their constituent molecules. This means that sound waves, as such, do not travel through the cochlear fluids. (The process resembles the Ramsauer–Townsend effect in particle collisions, wherein, a phase reversal of the partial wave results in the collision cross-section becoming almost zero.) It would otherwise be a disaster because the hum due to the Brownian motion would get mixed up with each incoming wave. This is a good teleological reason why the windows of the middle ear are where they are!

3. *Objective Values from Subjective Measurements*

In both the visual and auditory processes of measurement we have seen how the random element gives rise to differences in the actual values of measurement. The fluctuations are in the pupillary aperture in the first case, and in the density in cochlear fluids in the latter. In fact there are many other causes for fluctuations if one finds a need to go into details, and perhaps they may even be more important. But for a qualitative discussion, which is the best we can do, it would suffice to say that these random fluctuations are about a mean value, so that they have a normal distribution given by

$$\varphi_n(x) = \frac{n}{\sqrt{\pi}} \exp(-n^2 x^2)$$

where n is the index of precision. Now each particular reading $f_n(x)$ of an observable f at x is therefore the faltung of f with this error distribution:

$$f_n(x) = \int_{-\infty}^{\infty} f(x')\, \varphi_n(x - x')\, dx'$$

This gives a sequence of readings for the observable f. Now since φ_n tends to the Dirac δ-function as $n \to \infty$, it is obvious that $f_n(x) \to f(x)$ as $n \to \infty$. In other words, the objective reading is the limit of the sequence of measurements and not an 'exact' reading of the observable at the point in question.

Conclusion

A structure-preserving quantisation of a classical system necessarily implies to treat classical observables as averages over distributions, and not as objectively exact measurements (Shankara & Srinivas, 1971). This mathematical implication forces us to seek the statistical aspect of the whole process of measurement within the observer. This statistics could perhaps be identified with the error distribution generated by fluctuations in the observer, and in a crude approximation, it is assumed to have a Gaussian distribution. Since the limit of the Gaussian is the Dirac δ-measure, our faith that the average is really the objectively exact value, happens to be correct. But since, ultimately, all measurements are to be traced to the human observer, it appears as though it is legitimate to demand that a correct knowledge of all fluctuations in the physiology of vision and hearing is necessary to define a classical observable. The author believes that in the absence of this knowledge, which is, however, unnecessary for classical purposes, one fails to decide the corresponding quantum objects with the desired precision. But with the highly inadequate present-day physiological knowledge of vision and hearing, who can deny that this is undoubtedly a very severe demand?

Acknowledgment

I am grateful to Prof. E. C. G. Sudarshan for his advice in the preparation of this paper, to the two medical men Dr. H. Krishna Murthy and Dr. S. R. Chandrashekhar for interesting discussions and to Dr. S. R. Valluri, Director, National Aeronautical Laboratory, for his kind interest.

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